Calculus II: For Physical Sciences and Engineering (AS.110.109)

Lecture: 12

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Textbook Sections: 7.4

Todays Topic: Partial Fractions

Goal: Suppose we want to integrate a rational function:

$$f(x) = \frac{P(x)}{Q(x)} = \frac{p_n x^n + p_{n-1}x^{n-1} + \cdots + p_1 x + p_0}{q_m x^m + q_{m-1}x^{m-1} + \cdots + q_1 x + q_0} \qquad \begin{pmatrix} p_n \neq 0 \\ q_m \neq 0 \end{pmatrix}$$

In general, we would like to "split $f(x)$ up" into a sum of simpler-to-integrate functions.

Example: As we will see,

$$f(x) = \frac{x^2+2x-1}{2x^3+3x^2-2x} = \frac{1}{2}\cdot\frac{1}{x} + \frac{1}{5}\frac{1}{2x-1} - \frac{1}{10}\frac{1}{x+2}.$$

This makes in easy to integrate $f(x)$ (which would be quite difficult using the techniques we have seen so far):

$$\int f(x)\,dx = \frac{1}{2}\ln|x| + \frac{1}{10}\ln|2x-1| - \frac{1}{10}\ln|x+2| + C.$$

Without any further ado, here is the process for carrying this process out:

Step 1: If $\deg(P) = n \geq m = \deg(Q)$, then we first have to perform long division, and divide $P(x)$ by $Q(x)$ until a remainder $R(x)$ is obtained with $\deg(R) < \deg(Q)$:

$$f(x) = \frac{P(x)}{Q(x)} = S(x) + \frac{R(x)}{Q(x)}$$

Here, $S(x)$, $R(x)$ are both polynominals. This allows us to reduce our task to understanding proper rational functions: those with $\deg(P) < \deg(Q)$.

⚠ Skip this in lecture itself ⚠

A refresher on polynomial long division:

If this technique is unfamiliar to you, here is a guiding example that demonstrates the algorithm:

Example: Divide $(x^3 - 2x^2 - 4)$ by $(x-3)$

1) Write

$$x-3 \,\overline{)\, x^3 - 2x^2 + 0\cdot x - 4}$$

(notice we include the $0\cdot x$ term).

2) Divide the highest order term of the dividend by the highest order term of the divisor, and write the result in the corresponding spot above the bar (line up the powers):

$$\begin{array}{r} x^2 \\ x-3 \,\overline{)\, x^3 - 2x^2 + 0\cdot x - 4} \end{array}$$

3) Multiply the divisor by the result of Step 2, and write it under the dividend in the corresponding spot (i.e. line up the powers):

$$\begin{array}{r} x^2 \\ x-3 \,\overline{)\, x^3 - 2x^2 + 0\cdot x - 4} \\ x^3 - 3x^2 \end{array}$$

5) Subtract this new term from the dividend and write the remainder term to serve as the new dividend below like so:

$$\begin{array}{r} x^2 \\ x-3 \,\overline{)\, x^3 - 2x^2 + 0\cdot x - 4} \\ \underline{x^3 - 3x^2} \\ x^2 + 0\cdot x - 4 \end{array}$$

6) Repeat this process until the remainder has smaller power than the divisor:

$$\begin{array}{r} x^2 + x + 3 \\ x-3 \,\overline{)\, x^3 - 2x^2 + 0\cdot x - 4} \\ \underline{x^3 - 3x^2} \\ x^2 + 0\cdot x - 4 \\ \underline{x^2 - 3x} \\ 3x - 4 \\ \underline{3x - 9} \\ 5 \end{array}$$

$$\rightsquigarrow \quad \frac{x^3 - 2x^2 - 4}{x-3} = (x^2 + x + 3) + \frac{5}{x-3}$$

Step 2: Factor $Q(x)$ as far as possible:

Theorem: $Q(x)$ can always be factored into a product of

- linear factors: $(ax+b)$
- irreducible quadratics: (ax^2+bx+c) with $b^2-4ac<0$.

Example: Let $Q(x) = x^7+x^6-x-1$

Then one can show that

$$Q(x) = (x+1)^2(x-1)\cdot(x^2+x+1)(x^2-x+1).$$

In general, we can write

$$Q(x) = \underbrace{\left(\prod_{i=1}^{k}(a_ix+b_i)^{n_i}\right)}_{\text{all linear factors. Some can be repeated, so it could be that } n_i\geq 2).} \times \underbrace{\left(\prod_{j=1}^{\ell}(c_jx^2+d_jx+e_j)^{m_j}\right)}_{\text{all irred. quadratics. Again, some of these quadratics could appear multiple times, so that some } m_j \text{ might be } \geq 2.}$$

don't write in class!

Note: $(n_1+\cdots+n_k)+(m_1+\cdots+m_\ell)=\deg Q$.

Step 3: Express our proper rational function $f(x)=\frac{P(x)}{Q(x)}$ in the following expanded form: here all capital letters are currently unknown constants which will be determined in the next step:

$\frac{P(x)}{Q(x)} =$	For each:	Add the terms
terms corresponding to each distinct linear factor.	$\Rightarrow (ax+b)$	$\cdots \frac{A}{ax+b}$
$+$ terms corresponding to each repeated linear factor.	$\Rightarrow (ax+b)^n$	$\cdots \frac{A_1}{ax+b}+\frac{A_2}{(ax+b)^2}+\cdots+\frac{A_n}{(ax+b)^n}$
$+$ terms corresponding to each distinct quadratic factor.	$\Rightarrow (ax^2+bx+c)$	$\cdots \frac{Ax+B}{ax^2+bx+c}$
$+$ terms corresponding to each repeated quadratic factor.	$\Rightarrow (ax^2+bx+c)^n$	$\cdots \frac{A_1x+B_1}{ax^2+bx+c}+\frac{A_2x+B_2}{(ax^2+bx+c)^2}+\cdots+\frac{A_nx+B_n}{(ax^2+bx+c)^n}$

Example: For our example $Q(x) = x^7 + x^6 - x - 1$

$= (x+1)^2 (x-1) \cdot (x^2 + x + 1)(x^2 - x + 1)$,

Our expansion set up looks like:

$$\frac{P(x)}{Q(x)} = \frac{A}{x-1} + \frac{B_1}{x+1} + \frac{B_2}{(x+1)^2} + \frac{Cx+D}{x^2+x+1} + \frac{Ex+F}{x^2-x+1}$$

Step 4: Now we solve for the unknown coefficients (the capital letters):

(i) Cross multiply by $Q(x)$:

$$P(x) = Q(x) \cdot \{\text{expansion from Step 3}\}$$

Method 1:

(ii) Expand the RHS, and equate the corresponding coefficients of each power of x^k with those of $P(x)$

(iii) Solve this system of eqtns for the unknown coefficients.

Method 2: (better to illustrate this by example)

(ii) Only cancel factors in $Q(x)$ with the denominator in each term of the expansion, but don't expand beyond this

(iii) Plug in "good choices of x" to both sides to determine coefficients.

Example: For example take $Q(x) = x^3 + 2x^2 + 2x + 1$

$= (x+1)(x^2 + x + 1)$

and $P(x) = x^2 + 1$, with expansion

$$\frac{P(x)}{Q(x)} = \frac{A_1}{x+1} + \frac{B_1 x + B_2}{x^2+x+1},$$

we get $x^2 + 1 = P(x) = A_1 \cdot (x^2 + x + 1) + (B_1 x + B_2) \cdot (x+1)$.

Plug in:

$$\begin{cases} \text{(i) } x = 0 \implies 1 = A_1 + B_2 \\ \text{(ii) } x = 1 \implies 2 = 3A_1 + 2(B_1 + B_2) \\ \text{(iii) } x = -1 \implies 2 = A_1 \end{cases}$$

$$\begin{cases} \text{(i)} + \text{(iii)} \implies B_2 = -1 \\ \text{(i)} + \text{(ii)} + \text{(iii)} \implies B_1 = -1 \end{cases}$$

Thus, $$\frac{x^2+1}{x^3+2x^2+2x+1} = \frac{2}{x+1} - \frac{x+1}{x^2+x+1}.$$

Step 5: Integrate!

⚠ If you have an irred. quadratic, and can't to a u-sub, completing the square can help!

Example:
$$\int \frac{x^2+1}{x^3+2x^2+2x+1}\,dx = \int \frac{2}{x+1}\,dx - \int \frac{x+1}{x^2+x+1}\,dx$$
$$= 2\ln|x+1| - \frac{1}{2}\int \frac{2x+1}{x^2+x+1}\,dx - \frac{1}{2}\int \frac{1}{x^2+x+1}\,dx$$
$$= \ln(x+1)^2 - \ln\sqrt{x^2+x+1} - \frac{1}{2}\int \frac{1}{(x+\frac{1}{2})^2+\frac{3}{4}}\,dx$$

Note: $ax^2+bx+c = (\alpha x+\beta)^2+\delta \iff \begin{cases}\alpha=\sqrt{a}\\ \beta=\frac{b}{2\sqrt{a}}\end{cases}$ $\quad \delta = c-\frac{b^2}{4a}$

$$= \ln\left(\frac{x^2+2x+1}{\sqrt{x^2+x+1}}\right) - \frac{1}{2}\int \frac{1}{u^2+\frac{3}{4}}\,du \qquad \left[u = x+\tfrac{1}{2}\right]$$
$$= \ln\left(\frac{x^2+2x+1}{\sqrt{x^2+x+1}}\right) - 2\int \frac{1}{4u^2+3}\,du$$
$$= \ln\left(\frac{x^2+2x+1}{\sqrt{x^2+x+1}}\right) - \frac{1}{\sqrt{3}}\int \frac{\sec^2\theta}{(\tan^2\theta+1)}\,d\theta \qquad \left[\begin{array}{c} u = \frac{\sqrt{3}}{2}\tan\theta \\ \theta\in(-\frac{\pi}{2},\frac{\pi}{2}) \\ du = \frac{\sqrt{3}}{2}\sec^2\theta\,d\theta\end{array}\right]$$
$$= \ln\left(\frac{x^2+2x+1}{\sqrt{x^2+x+1}}\right) - \frac{1}{\sqrt{3}}\theta + C$$
$$= \ln\left(\frac{x^2+2x+1}{\sqrt{x^2+x+1}}\right) - \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{2}{\sqrt{3}}\left(x+\frac{1}{2}\right)\right) + C.$$